Finite-Region Stabilization via Dynamic Output Feedback for 2-D Roesser Models

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Abstract: Finite-region stability (FRS), a generalization of finite-time stability (FTS), has been used to analyze the transient behavior of discrete two-dimensional (2-D) systems. In this paper, we consider the problem of FRS for discrete 2-D Roesser models via dynamic output feedback. First, a sufficient condition is given to design the dynamic output feedback controller with a state feedback-observer structure, which ensures the closed-loop system FRS. Then, this condition is reducible to a condition that is solvable by linear matrix inequalities (LMIs). Finally, viable experimental results are demonstrated by an illustrative example.
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1 Introduction

The two-dimensional (2-D) state-space theory was first introduced by Roesser [1]. Since then, 2-D systems have been widely studied in [2–7] with a proliferation of emerging applications over the last few decades, including image processing, decoding, encoding, iterative learning, repetitive processes and so forth. As such, the research on 2-D systems has been a hot area in control field. Roesser model as one of the commonly used models of 2-D systems has attracted much attention of many researchers, and many interesting findings on stability and control have been obtained [8–13]. For example, Lam *et al.* [8] investigated the stabilization problem for uncertain 2-D Roesser model via dynamic output feedback. Nachidi *et al.* [9] designed the static output feedback controller for 2-D Roesser models. Results on $l_2 - l_\infty$ stability analysis were established for a class of 2-D nonlinear disturbed systems [10], which guarantees asymptotic stability without external interference. Besides, Ahn *et al.* [13] solved the problems of dissipative control and filtering for 2-D systems, by providing a sufficient condition to check asymptotic stability and 2-D $(Q, S, R) - \alpha$ dissipativity. However, these results on stability or control were associated with Lyapunov asymptotic stability (LAS).

Apart from LAS, recent years have also witnessed growing interests on finite-time stability (FTS) for one-dimensional (1-D) systems. The concept of FTS was first introduced in [14], and reintroduced by Dorata in [15] which is related to dynamical systems whose state does not exceed some bounds during the specified time interval. It is important to note that FTS and LAS are completely independent concepts. FTS aims at analyzing transient behavior of a system

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within the finite interval time rather than the asymptotic behavior within the infinite time. FTS plays vital roles in many practical applications, for instance the problem of not exceeding some given bounds for the state trajectories, when there exist some saturation elements in the control loop; or the problem of controlling the trajectory of a spacecraft from an initial point to a final point in a prescribed time interval. With the in-depth of research on FTS theory for 1-D systems, many interesting results have come into play, see [16–23]. Among them, Amato *et al.* [21] used a two-step procedure (state feedback design followed by observer synthesis) in the finite-time stabilization problem for 1-D continuous-time linear systems.

In addition, the study of the finite region for discrete 2-D systems have been popping up. In [24, 25], the concept of finite-region stability (FRS) as the extension of FTS in 1-D systems case to discrete 2-D systems was put forward, and finite-region stabilization of these discrete 2-D models in the state feedback case was investigated. In practice, the system state is often unknown or cannot be directly measured. Therefore, it is necessary to study the dynamic output feedback stabilization problem, and consider designing an observer to estimate the state. For 1-D systems, the problem of observer-based dynamic output feedback is challenging because of the coupling of the observer design and the controller design, not to mention 2-D systems.

In this paper, motivated by literature [21], we focus on the finite-region stabilization for discrete 2-D Roesser models via dynamic output feedback. We first introduce a Luenberger observer with a state feedback controller, which is a special dynamic output feedback controller. Then we get a closed-loop system that treats the state estimation errors as external perturbations, and the boundedness condition of the external perturbations can be guaranteed by Theorem 3.1 in [25]. In this way, the problem of finite-region stabilization for discrete 2-D Roesser models via dynamic output feedback is converted into the problem of designing an observer to guarantee the closed-loop system finite-region boundedness (FRB). Furthermore, we give a generic sufficient condition and a sufficient condition that is solvable by linear matrix inequalities (LMIs) for the existence of such an dynamic output feedback controller that guarantees the closed-loop system to be FRS.

Notations N^+ denotes a set of positive integers, R^n is the n-dimensional space with inner product x^Ty. $A > 0$ means that the matrix A is symmetric positive definite. A^T denotes the transpose of matrix A, I represents the identity matrix. $\lambda(A)$ denotes the eigenvalue of A, $\lambda_{\max}(A)$ is the maximum eigenvalue of A and $\lambda_{\min}(A)$ is the minimum eigenvalue of A. $*$ represents the symmetric terms in a matrix.

2 Preliminaries and problem statement

In this paper we consider the following 2-D discrete-time linear system in the Roesser model:

$$x^+(i,j) = Ax(i,j) + Bu(i,j), \quad x_0(i,j), \tag{1}$$

$$y(i,j) = Cx(i,j), \tag{2}$$

where $x(i,j) = \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} \in R^n$ is the state vector, $x^+(i,j) = \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix}$, $u(i,j) \in R^p$ is the 2-D control input vector, and $y(i,j) \in R^q$ is the 2-D output vector, $x_0(i,j) =$

$\begin{bmatrix} x^h(0,j) \\ x^v(i,0) \end{bmatrix}$ is the boundary condition, i, j are the horizontal and vertical discrete variables; $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ and $C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$ are constant real matrices with appropriate dimensions.

Define the finite-region for discrete 2-D Roesser model (1)-(2) as follows:

$$I_0 = I_1 \times I_2 = \{(i,j) | 0 \leq i \leq I_1, 0 \leq j \leq I_2; I_1, I_2 \in N^+\}. \tag{3}$$

For 1-D discrete systems, the concept of FTS is given below.

Definition 2.1. *[20] Given three positive scalars c_1, c_2, M, with $0 < c_1 < c_2$, $M \in N^+$, and a positive definite matrix R, the discrete-time linear system*

$$x(k+1) = Ax(k), \quad x(0) = x_0$$

is said to be FTS with respect to (c_1, c_2, M, R), if

$$x_0^T R x_0 \leq c_1 \Rightarrow x^T(k) R x(k) < c_2, \quad k \in \{1, \cdots, M\}.$$

It is worth noting that the concept of finite-time boundedness (FTB) was first given in [26, 27] when one deals with the FTS of 1-D linear systems in the presence of nonzero exogenous perturbations.

The definitions of FRS and FRB for 2-D discrete systems given in [25] are different from those of FTS and FTB for 1-D systems. Thus, we slightly change the definitions of FRS and FRB for 2-D systems, to keep their forms consistent with the definitions of FTS and FTB for 1-D systems.

Definition 2.2. *Given two positive scalars c_1, c_2, with $c_1 < c_2$, I_0, where I_0 is defined in (3), and positive definite matrix R, where $R = diag\{R_1, R_2\}$, $R_1 > 0, R_2 > 0$, the system (1) with zero input:*

$$x^+(i,j) = Ax(i,j), \quad x_0(i,j) \tag{4}$$

is said to be FRS with respect to (c_1, c_2, I_0, R), if

$$x_0^T(i,j) R x_0(i,j) \leq c_1 \Rightarrow x^T(i,j) R x(i,j) < c_2, \quad \forall (i,j) \in I_0.$$

Considering that the exogenous perturbations influence system (4), further, we introduce the following system

$$x^+(i,j) = Ax(i,j) + Gw(i,j), \quad x_0(i,j). \tag{5}$$

As usual, we impose the following restrictions on exogenous perturbations.

Assumption 2.1 Assume that the external perturbation $w(i,j)$ of system (5) satisfies the following condition:

$$\exists\, d > 0 \;\; s.t. \;\; w^T(i,j) R w(i,j) < d, \quad \forall (i,j) \in I_0. \tag{6}$$

Definition 2.3. *Given three positive scalars c_1, c_2, d, with $c_1 < c_2$, I_0, where I_0 is defined in (3), and positive definite matrix R, where $R = diag\{R_1, R_2\}$, $R_1 > 0, R_2 > 0$, the 2-D Roesser model (5) is said to be FRB with respect to (c_1, c_2, I_0, R, d), if*

$$x_0^T(i,j)Rx_0(i,j) \le c_1 \Rightarrow x^T(i,j)Rx(i,j) < c_2, \quad \forall (i,j) \in I_0,$$

for all $w(i,j)$ satisfying Assumption 2.1.

In this paper, we will study the finite-region stabilization issue for discrete 2-D Roesser models via dynamic output feedback. First consider the general dynamic output feedback controller of the given discrete 2-D system (1)-(2):

$$\xi^+(i,j) = A_c\xi(i,j) + B_c y(i,j), \quad \xi_0(i,j) = 0, \tag{7}$$

$$u(i,j) = C_c\xi(i,j) + D_c y(i,j), \tag{8}$$

where $\xi^+(i,j) = \begin{bmatrix} \xi^h(i+1,j) \\ \xi^v(i,j+1) \end{bmatrix}$, $\xi(i,j) = \begin{bmatrix} \xi^h(i,j) \\ \xi^v(i,j) \end{bmatrix}$, $\xi_0(i,j) = \begin{bmatrix} \xi^h(0,j) \\ \xi^v(i,0) \end{bmatrix}$, and $A_c = \begin{bmatrix} A_{c,11} & A_{c,12} \\ A_{c,21} & A_{c,22} \end{bmatrix}$, $B_c = \begin{bmatrix} B_{c,1} \\ B_{c,2} \end{bmatrix}$, $C_c = [C_{c,1}, C_{c,2}]$ and D_c are constant real matrices with appropriate dimensions.

Together with the system (1)-(2) and the controller (7)-(8), then

$$x^+(i,j) = (A + BD_cC)x(i,j) + BC_c\xi(i,j), \quad x_0(i,j), \tag{9}$$

$$\xi^+(i,j) = B_cCx(i,j) + A_c\xi(i,j), \quad \xi_0(i,j) = 0. \tag{10}$$

Remark 2.1 Systems (9)-(10) are well posed. Given controller (7)-(8), for any initial condition $x_0(i,j)$ with $x_0^T(i,j)Rx_0(i,j) \le c_1$ and $\xi_0(i,j) = 0$, $\xi(i,j)$ is unique and it makes sense to let $\xi^T(i,j)R\xi(i,j) < d$.

Clearly, the problem of finite-region stabilization for system (1)-(2) via dynamic output feedback is now simplified to the FRB problem of system (9). And this issue can be specifically described as the following problem.

Problem 2.1 Given three positive scalars c_1, c_2, d, with $c_1 < c_2$, I_0, where I_0 is defined in (3), and a positive definite matrix R, where $R = \text{diag}\{R_1, R_2\}$, $R_1 > 0, R_2 > 0$, our goral is to find a dynamic output feedback controller in the form (7)-(8) such that 2-D discrete system (1)-(2) under the input (8), *i*.e. system (9), is FRB with respect to (c_1, c_2, I_0, R, d).

In general, it is quite difficult to design a generic dynamic output feedback controller (7)-(8) for discrete system (1)-(2). Fortunately, a two-step procedure for designing a dynamic output feedback controller of 1-D linear systems has been proposed in [21]. In light of this, we next design a specific one for discrete 2-D Roesser model. The existence of such a controller ensuring the closed-loop system FRS can be studied by using finite-region stabilization via state feedback.

First, we introduce the Luenberger observer [28] of system (1)-(2):

$$\xi^+(i,j) = A\xi(i,j) + Bu(i,j) + L\left(C\xi(i,j) - y(i,j)\right), \quad \xi_0(i,j) = 0, \tag{11}$$

where $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ is the observer gain matrix with approximate dimensions.

Now we feedback the state estimation via the state feedback controller

$$u(i,j) = K\xi(i,j). \tag{12}$$

Note that when the state feedback controller K exists, it can be designed by employing the method given in Theorem 3.3 of [25]. Therefore, it is reasonable to make the following assumption.

Assumption 2.2 There exists a state feedback controller $u(i,j) = Kx(i,j)$ such that the closed-loop control system of discrete 2-D system (1) is FRS, where $K = [K_1, K_2]$.

If such an observer gain L in (11) exists, the corresponding controller (11)-(12) is a dynamic output feedback controller (7)-(8), where $A_c = A + BK + LC$, $B_c = -L$, $C_c = K$ and $D_c = 0$, then the closed-loop state equations (9)-(10) become the following form

$$x^+(i,j) = Ax(i,j) + BK\xi(i,j), \quad x_0(i,j), \tag{13}$$

$$\xi^+(i,j) = -LCx(i,j) + (A + BK + LC)\xi(i,j), \quad \xi_0(i,j) = 0. \tag{14}$$

Let the state estimation error be $e(i,j) = x(i,j) - \xi(i,j)$, then the state equations (13)-(14) can be translated into the form based on estimation error

$$x^+(i,j) = (A + BK)x(i,j) - BKe(i,j), \quad x_0(i,j), \tag{15}$$

with

$$e^+(i,j) = (A + LC)e(i,j), \quad e_0(i,j) = x_0(i,j). \tag{16}$$

Therefore, the system state evolution can be codetermined by the matrix $A + BK$ and the behavior of external input $e(i,j)$. If $e(i,j) = 0$, it follows from Assumption 2.2 that the system (15) is FRS. If $e(i,j) \neq 0$, there exists inaccurate state estimation, and the existence of a nonzero estimation error may destroy the FRS of closed-loop system $x^+(i,j) = (A+BK)x(i,j)$ obtained by the state feedback controller. In this case, we need to study the FRB problem of system (15), which treats error-term system (16) as external perturbations. Note that if the error-term system (16) is FRS in given finite-region I_0, then for given two constants c_1, d, with $c_1 < d$, a positive definite matrix R, and the initial condition $x_0^T(i,j)Rx_0(i,j) \leq c_1$, the error $e(i,j)$ satisfies $e^T(i,j)Re(i,j) < d$.

Based on the above discussion, our goal is to design an observer gain L in (11) such that the system (15) is FRB for all admissible estimation error (16). To summarize, Problem 2.1 can boil down to the following problem.

Problem 2.2 When there exists a finite-region stabilizable system (1) via state feedback, we find an observer gain L such that system (15) is FRB with respect to (c_1, c_2, I_0, R, d).

3 Main results

The following theorems give the sufficient conditions for the solvability of Problem 2.2.

Theorem 3.1. *Given system (15)-(16) and three positive scalars c_1, c_2, d, with $c_1 < c_2, c_1 < d$, there exist nonnegative scalars $0 < \eta < 1$, α_l, symmetric positive definite matrices P_l, Q_l, and symmetric matrices M_l, where $l = 1, 2$, such that* *i)*

$$\begin{bmatrix} -\alpha_1 P^- & 0 & 0 & {A_1}^T P_1 + K^T {B_1}^T P_1 \\ * & -\alpha_1 Q^- & {A_1}^T Q_1 + C^T M_1 & -K^T {B_1}^T P_1 \\ * & * & -Q_1 & 0 \\ * & * & * & -P_1 \end{bmatrix} < 0, \tag{17}$$

$$\begin{bmatrix} -\alpha_2 P_- & 0 & 0 & {A_2}^T P_2 + K^T {B_2}^T P_2 \\ * & -\alpha_2 Q_- & {A_2}^T Q_2 + C^T M_2 & -K^T {B_2}^T P_2 \\ * & * & -Q_2 & 0 \\ * & * & * & -P_2 \end{bmatrix} < 0, \tag{18}$$

where $A_1 = [A_{11}, A_{12}]$*,* $A_2 = [A_{21}, A_{22}]$*,* $P^- = \begin{bmatrix} P_1 & 0 \\ * & \frac{1}{c_2} P_2 \end{bmatrix}$*,* $Q^- = \begin{bmatrix} Q_1 & 0 \\ * & \frac{1}{d} Q_2 \end{bmatrix}$*,* $P_- = \begin{bmatrix} \frac{1}{c_2} P_1 & 0 \\ * & P_2 \end{bmatrix}$ *and* $Q_- = \begin{bmatrix} \frac{1}{d} Q_1 & 0 \\ * & Q_2 \end{bmatrix}$.

ii)

$$\frac{\alpha_0 \eta c_1 \left(\lambda_{max}(\widetilde{P}_1) + \lambda_{max}(\widetilde{Q}_1) \right) + I_1 \alpha_0 (1-\eta) \left(\lambda_{max}(\widetilde{P}_2) + \lambda_{max}(\widetilde{Q}_2) \right)}{\lambda_{min}(\widetilde{P}_1)} < \eta c_2, \tag{19}$$

$$\frac{\alpha_0 (1-\eta) c_1 \left(\lambda_{max}(\widetilde{P}_2) + \lambda_{max}(\widetilde{Q}_2) \right) + I_2 \alpha_0 \eta \left(\lambda_{max}(\widetilde{P}_1) + \lambda_{max}(\widetilde{Q}_1) \right)}{\lambda_{min}(\widetilde{P}_2)} < (1-\eta) c_2, \tag{20}$$

where $\alpha_0 = max\{1, {\alpha_1}^{I_1}, {\alpha_2}^{I_2}\}$*,* $\widetilde{P}_l = R_l^{-\frac{1}{2}} P_l R_l^{-\frac{1}{2}}$*,* $\widetilde{Q}_l = R_l^{-\frac{1}{2}} Q_l R_l^{-\frac{1}{2}}$*,* $l = 1, 2$.

In this case, the discrete system (15) is FRB with respect to (c_1, c_2, I_0, R, d)*, and the dynamic output feedback controller which makes the system (1)-(2) FRS has the structure (11)-(12) with* $L = Q^{-1} M$*, where* $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$*,* $Q = \begin{bmatrix} Q_1 & 0 \\ * & Q_2 \end{bmatrix}$*, and* $M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}$.

Proof. First, using Schur complement lemma, we produce an equivalent form of (17) and (18).

By Schur complement lemma [29], the condition (17) is equivalent to

$$\begin{bmatrix} (A_1 + B_1 K)^T P_1 (A_1 + B_1 K) - \alpha_1 P^- & -(A_1 + B_1 K)^T P_1 B_1 K & 0 \\ * & (B_1 K)^T P_1 B_1 K - \alpha_1 Q^- & (Q_1 A_1 + M_1 C)^T \\ * & * & -Q_1 \end{bmatrix} < 0. \tag{21}$$

Re-applying Schur complement lemma [29] to (21) produces

$$\begin{bmatrix} \begin{matrix} (A_1 + B_1 K)^T P_1 (A_1 + B_1 K) - \alpha_1 P^- \\ -(B_1 K)^T P_1 (A_1 + B_1 K) \end{matrix} & \begin{matrix} -(A_1 + B_1 K)^T P_1 B_1 K \\ (B_1 K)^T P_1 B_1 K + (Q_1 A_1 + M_1 C)^T {Q_1}^{-1} (Q_1 A_1 + M_1 C) - \alpha_1 Q^- \end{matrix} \end{bmatrix} < 0. \tag{22}$$

Similarly, applying Schur complement lemma [29] twice to the condition (18) yields

$$\begin{bmatrix} (A_2+B_2K)^TP_2(A_2+B_2K)-\alpha_2P_- \\ -(B_2K)^TP_2(A_2+B_2K) \end{bmatrix.$$

$$\left. \begin{matrix} -(A_2+B_2K)^TP_2B_2K \\ (B_2K)^TP_2B_2K+(Q_2A_2+M_2C)^TQ_2{}^{-1}(Q_2A_2+M_2C)-\alpha_2Q_- \end{matrix} \right] < 0. \tag{23}$$

Let $M_1 = Q_1L_1$ and $M_2 = Q_2L_2$, the conditions (22)-(23) can be rewritten as

$$\left[\begin{matrix} (A_1+B_1K)^TP_1(A_1+B_1K)-\alpha_1P^- \\ -(B_1K)^TP_1(A_1+B_1K) \end{matrix} \right.$$

$$\left. \begin{matrix} -(A_1+B_1K)^TP_1B_1K \\ (B_1K)^TP_1B_1K+(A_1+L_1C)^TQ_1(A_1+L_1C)-\alpha_1Q^- \end{matrix} \right] < 0, \tag{24}$$

$$\left[\begin{matrix} (A_2+B_2K)^TP_2(A_2+B_2K)-\alpha_2P_- \\ -(B_2K)^TP_2(A_2+B_2K) \end{matrix} \right.$$

$$\left. \begin{matrix} -(A_2+B_2K)^TP_2B_2K \\ (B_2K)^TP_2B_2K+(A_2+L_2C)^TQ_2(A_2+L_2C)-\alpha_2Q_- \end{matrix} \right] < 0. \tag{25}$$

Second, we derive the recursive relations of the weights of state variables.

For simplicity, let $z^+(i,j) = \begin{bmatrix} x^+(i,j) \\ e^+(i,j) \end{bmatrix}$, $z(i,j) = \begin{bmatrix} x(i,j) \\ e(i,j) \end{bmatrix}$, and $z_0(i,j) = \begin{bmatrix} x_0(i,j) \\ x_0(i,j) \end{bmatrix}$. Then, the system (15)-(16) reduces to

$$z^+(i,j) = \begin{bmatrix} A+BK & -BK \\ 0 & A+LC \end{bmatrix} z(i,j), \quad z_0(i,j). \tag{26}$$

Next, we define the Lyapunov functions of system (26) as follows

$$V_1(z^h(i,j)) = z^{hT}(i,j) \begin{bmatrix} P_1 & 0 \\ * & Q_1 \end{bmatrix} z^h(i,j),$$
$$V_2\left(z^v(i,j)\right) = z^{vT}(i,j) \begin{bmatrix} P_2 & 0 \\ * & Q_2 \end{bmatrix} z^v(i,j),$$

where $z^h(i,j) = \begin{bmatrix} x^h(i,j) \\ e^h(i,j) \end{bmatrix}$, $z^v(i,j) = \begin{bmatrix} x^v(i,j) \\ e^v(i,j) \end{bmatrix}$. Then it follows that

$$\begin{aligned}
& V_1(z^h(i+1,j)) - \alpha_1 V_1(z^h(i,j)) - \alpha_1 {z^v}^T(i,j) \begin{bmatrix} \frac{1}{c_2}P_2 & 0 \\ * & \frac{1}{d}Q_2 \end{bmatrix} z^v(i,j) \\
=& {z^h}^T(i+1,j) \begin{bmatrix} P_1 & 0 \\ * & Q_1 \end{bmatrix} z^h(i+1,j) - \alpha_1 {z^h}^T(i,j) \begin{bmatrix} P_1 & 0 \\ * & Q_1 \end{bmatrix} z^h(i,j) - \alpha_1 {z^v}^T(i,j) \\
& \begin{bmatrix} \frac{1}{c_2}P_2 & 0 \\ * & \frac{1}{d}Q_2 \end{bmatrix} z^v(i,j) \\
=& z^T(i,j) \begin{bmatrix} (A_1+B_1K)^T & 0 \\ -K^T {B_1}^T & (A_1+L_1C)^T \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ * & Q_1 \end{bmatrix} \begin{bmatrix} A_1+B_1K & -B_1K \\ 0 & A_1+L_1C \end{bmatrix} z(i,j) \\
& - z^T(i,j) \begin{bmatrix} \alpha_1 P^- & 0 \\ * & \alpha_1 Q^- \end{bmatrix} z(i,j) \\
=& z^T(i,j) \begin{bmatrix} (A_1+B_1K)^T P_1 (A_1+B_1K) - \alpha_1 P^- & -(A_1+B_1K)^T P_1 B_1 K \\ -K^T {B_1}^T P_1 (A_1+B_1K) & (B_1K)^T P_1 B_1 K + (A_1+L_1C)^T Q_1 (A_1+L_1C) - \alpha_1 Q^- \end{bmatrix} z(i,j), \qquad (27)
\end{aligned}$$

where $A_1 = [A_{11}, A_{12}]$, $P^- = \begin{bmatrix} P_1 & 0 \\ * & \frac{1}{c_2}P_2 \end{bmatrix}$, $Q^- = \begin{bmatrix} Q_1 & 0 \\ * & \frac{1}{d}Q_2 \end{bmatrix}$.

Similarly, we can obtain the following equation

$$\begin{aligned}
& V_2(z^v(i,j+1)) - \alpha_2 V_2(z^v(i,j)) - \alpha_2 {z^h}^T(i,j) \begin{bmatrix} \frac{1}{c_2}P_1 & 0 \\ * & \frac{1}{d}Q_1 \end{bmatrix} z^h(i,j) \\
=& z^T(i,j) \begin{bmatrix} (A_2+B_2K)^T P_2 (A_2+B_2K) - \alpha_2 P_- & -(A_2+B_2K)^T P_2 B_2 K \\ -K^T {B_2}^T P_2 (A_2+B_2K) & (B_2K)^T P_2 B_2 K + (A_2+L_2C)^T Q_2 (A_2+L_2C) - \alpha_2 Q_- \end{bmatrix} z(i,j), \qquad (28)
\end{aligned}$$

where $A_2 = [A_{21}, A_{22}]$, $P_- = \begin{bmatrix} \frac{1}{c_2}P_1 & 0 \\ * & P_2 \end{bmatrix}$ and $Q_- = \begin{bmatrix} \frac{1}{d}Q_1 & 0 \\ * & Q_2 \end{bmatrix}$.

The condition (24) implies that, for all $(i,j) \in I_0$ and $e^T(i,j)Re(i,j) < d$, (27) < 0, that is

$$V_1(z^h(i+1,j)) < \alpha_1 V_1(z^h(i,j)) + \alpha_1 {z^v}^T(i,j) \begin{bmatrix} \frac{1}{c_2}P_2 & 0 \\ * & \frac{1}{d}Q_2 \end{bmatrix} z^v(i,j). \qquad (29)$$

Similarly, it follows from (25) and (28) that

$$V_2(z^v(i,j+1)) < \alpha_2 V_2(z^v(i,j)) + \alpha_2 {z^h}^T(i,j) \begin{bmatrix} \frac{1}{c_2}P_1 & 0 \\ * & \frac{1}{d}Q_1 \end{bmatrix} z^h(i,j). \qquad (30)$$

For (29) and (30) iterations, we have

$$
\begin{aligned}
V_1(z^h(i,j)) <& \alpha_1 V_1(z^h(i-1,j)) + \alpha_1 z^{vT}(i-1,j) \begin{bmatrix} \frac{1}{c_2}P_2 & 0 \\ * & \frac{1}{d}Q_2 \end{bmatrix} z^v(i-1,j) \\
<& {\alpha_1}^2 V_1(z^h(i-2,j)) + \sum_{k=1}^{2} {\alpha_1}^k z^{vT}(i-k,j) \begin{bmatrix} \frac{1}{c_2}P_2 & 0 \\ * & \frac{1}{d}Q_2 \end{bmatrix} z^v(i-k,j) \\
<& {\alpha_1}^i V_1(z^h(0,j)) + \sum_{k=1}^{i} {\alpha_1}^k z^{vT}(i-k,j) \begin{bmatrix} \frac{1}{c_2}P_2 & 0 \\ * & \frac{1}{d}Q_2 \end{bmatrix} z^v(i-k,j) \\
=& {\alpha_1}^i z^{hT}(0,j) \begin{bmatrix} P_1 & 0 \\ * & Q_1 \end{bmatrix} z^h(0,j) + \sum_{k=1}^{i} {\alpha_1}^k z^{vT}(i-k,j) \begin{bmatrix} \frac{1}{c_2}P_2 & 0 \\ * & \frac{1}{d}Q_2 \end{bmatrix} z^v(i-k,j) \\
\leq& {\alpha_1}^i \left(\lambda_{\max}(\widetilde{P}_1) + \lambda_{\max}(\widetilde{Q}_1) \right) x^{hT}(0,j) R_1 x^h(0,j) + \sum_{k=1}^{i} {\alpha_1}^k \left(\frac{1}{c_2} \lambda_{\max}(\widetilde{P}_2) \right. \\
& \left. x^{vT}(i-k,j) R_2 x^v(i-k,j) + \frac{1}{d} \lambda_{\max}(\widetilde{Q}_2) e^{vT}(i-k,j) R_2 e^v(i-k,j) \right). \qquad (31)
\end{aligned}
$$

Based on Theorem 3.2 in [25], we obtain that $e^{hT}(i,j)R_1e^h(i,j) < \eta d$ and $e^{vT}(i,j)R_2e^v(i,j) < (1-\eta)d$ when initial condition $x^h(0,j)$ satisfies $x^{hT}(0,j)R_1x^h(0,j) \leq \eta c_1 < \eta d$, where $0 < \eta < 1$. Therefore, inequality (31) is translated into

$$
\begin{aligned}
V_1(z^h(i,j)) <& \alpha_0 \eta c_1 \left(\lambda_{\max}(\widetilde{P}_1) + \lambda_{\max}(\widetilde{Q}_1) \right) + I_1 \alpha_0 \left(\frac{1}{c_2} \lambda_{\max}(\widetilde{P}_2) x^{vT}(i-k,j) R_2 x^v(i-k,j) \right. \\
& \left. + (1-\eta) \lambda_{\max}(\widetilde{Q}_2) \right), \qquad (32)
\end{aligned}
$$

where $\alpha_0 = \max\{1, {\alpha_1}^{I_1}, {\alpha_2}^{I_2}\}$.

On the other hand,

$$
\begin{aligned}
V_1\left(z^h(i,j)\right) \geq& \lambda_{\min}(\widetilde{P}_1) x^{hT}(i,j) R_1 x^h(i,j) + \lambda_{\min}(\widetilde{Q}_1) e^{hT}(i,j) Q_1 e^h(i,j) \\
>& \lambda_{\min}(\widetilde{P}_1) x^{hT}(i,j) R_1 x^h(i,j). \qquad (33)
\end{aligned}
$$

From (32) and (33), we have

$$
\begin{aligned}
x^{hT}(i,j) R_1 x^h(i,j) <& \frac{I_1 \alpha_0 \left(\frac{1}{c_2} \lambda_{\max}(\widetilde{P}_2) x^{vT}(i-k,j) R_2 x^v(i-k,j) + (1-\eta) \lambda_{\max}(\widetilde{Q}_2) \right)}{\lambda_{\min}(\widetilde{P}_1)} \\
& + \frac{\alpha_0 \eta c_1 \left(\lambda_{\max}(\widetilde{P}_1) + \lambda_{\max}(\widetilde{Q}_1) \right)}{\lambda_{\min}(\widetilde{P}_1)}. \qquad (34)
\end{aligned}
$$

Using the same method for $V_2(z^v(i,j))$, we have

$$x^{vT}(i,j)R_2x^v(i,j) < \frac{I_2\alpha_0\left(\frac{1}{c_2}\lambda_{\max}(\widetilde{P}_1)x^{hT}(i,j-l)R_1x^h(i,j-l)+\eta\lambda_{\max}(\widetilde{Q}_1)\right)}{\lambda_{\min}(\widetilde{P}_2)} + \frac{\alpha_0(1-\eta)c_1\left(\lambda_{\max}(\widetilde{P}_2)+\lambda_{\max}(\widetilde{Q}_2)\right)}{\lambda_{\min}(\widetilde{P}_2)}. \tag{35}$$

Finally, we will use mathematical induction to prove the following conclusion: for given $j \in I_2$, if $x^{hT}(0,j)R_1x^h(0,j) \leq \eta c_1 < \eta c_2$, $0 < \eta < 1$, there exist two inequalities

$$x^{hT}(i,j)R_1x^h(i,j) < \eta c_2, \tag{36}$$

$$x^{vT}(i,j)R_2x^v(i,j) < (1-\eta)c_2. \tag{37}$$

Setting $i = 0$ in (35), we have

$$\begin{aligned} &x^{vT}(0,j)R_2x^v(0,j) \\ &< \frac{I_2\alpha_0\eta\left(\lambda_{\max}(\widetilde{P}_1)+\lambda_{\max}(\widetilde{Q}_1)\right)+\alpha_0(1-\eta)c_1\left(\lambda_{\max}(\widetilde{P}_2)+\lambda_{\max}(\widetilde{Q}_2)\right)}{\lambda_{\min}(\widetilde{P}_2)}. \end{aligned} \tag{38}$$

It follows from condition (20) that

$$x^{vT}(0,j)R_2x^v(0,j) < (1-\eta)c_2. \tag{39}$$

Setting $i = 1$ in (34) and using (39) and the condition (19), we have

$$\begin{aligned} x^{hT}(1,j)R_1x^h(1,j) <& \frac{I_1\alpha_0(1-\eta)\left(\lambda_{\max}(\widetilde{P}_2)+\lambda_{\max}(\widetilde{Q}_2)\right)+\alpha_0\eta c_1\left(\lambda_{\max}(\widetilde{P}_1)+\lambda_{\max}(\widetilde{Q}_1)\right)}{\lambda_{\min}(\widetilde{P}_1)} \\ <& \eta c_2. \end{aligned} \tag{40}$$

It is easy to obtain from both (40) and condition (19) that

$$x^{vT}(1,j)R_2x^v(1,j) < (1-\eta)c_2. \tag{41}$$

Suppose that the result (36) holds for $0 \leq i \leq I_1 - 1$. By direct calculation, we can obtain that $x^{vT}(i,j)R_2x^v(i,j) < (1-\eta)c_2$ also holds for any $0 \leq i \leq I_1 - 1$. Next we only need to prove that $x^{hT}(I_1,j)R_1x^h(I_1,j) < \eta c_2$ and $x^{vT}(I_1,j)R_2x^v(I_1,j) < (1-\eta)c_2$.

For fixed $i = I_1$, we have

$$
\begin{aligned}
{x^h}^T(I_1,j)R_1x^h(I_1,j) <& \frac{I_1\alpha_0\left(\frac{1}{c_2}\lambda_{\max}(\widetilde{P}_2)x^{vT}(I_1-k,j)R_2x^v(I_1-k,j)+(1-\eta)\lambda_{\max}(\widetilde{Q}_2)\right)}{\lambda_{\min}(\widetilde{P}_1)}\\
&+\frac{\alpha_0\eta c_1\left(\lambda_{\max}(\widetilde{P}_1)+\lambda_{\max}(\widetilde{Q}_1)\right)}{\lambda_{\min}(\widetilde{P}_1)}\\
<&\frac{\alpha_0\eta c_1\left(\lambda_{\max}(\widetilde{P}_1)+\lambda_{\max}(\widetilde{Q}_1)\right)+I_1\alpha_0(1-\eta)\left(\lambda_{\max}(\widetilde{P}_2)+\lambda_{\max}(\widetilde{Q}_2)\right)}{\lambda_{\min}(\widetilde{P}_1)}\\
<&\eta c_2, \qquad (42)
\end{aligned}
$$

and

$$x^{vT}(I_1,j)R_2x^v(I_1,j) < (1-\eta)c_2.$$

Hence, for any $(i,j)\in I_0$, the results (36) and (37) are satisfied.

Therefore,

$$x^T(i,j)Rx(i,j) = {x^h}^T(i,j)R_1x^h(i,j)+x^{vT}(i,j)R_2x^v(i,j) < c_2$$

for all $(i,j)\in I_0$. This implies that the discrete 2-D system (15) is FRB with respect to $(c_1.c_2, I_0, R, d)$, and the proof is completed. □

It should be pointed out that the conditions ii) in Theorem 3.1 are not LMIs conditions. To use LMI toolbox of Matlab to find the feasible solution, we impose additional conditions on the conditions ii) in Theorem 3.1, which produces the following LMIs based feasibility problems

$$\lambda_{l1}I < \widetilde{P}_l < \lambda_{l2}I, \qquad 0<\widetilde{Q}_l<\lambda_{l3}I, \qquad l=1,2, \tag{43}$$

$$\alpha_0\eta c_1\left(\lambda_{12}+\lambda_{13}\right)+I_1\alpha_0(1-\eta)\left(\lambda_{22}+\lambda_{23}\right)<\eta c_2\lambda_{11}, \tag{44}$$

$$\alpha_0(1-\eta)c_1\left(\lambda_{22}+\lambda_{23}\right)+I_2\alpha_0\eta\left(\lambda_{12}+\lambda_{13}\right)<(1-\eta)c_2\lambda_{21}, \tag{45}$$

where $\lambda_{l1},\lambda_{l2},\lambda_{l3}$ are positive numbers.

It is easy to verify LMIs conditions (43)-(45) can guarantee the conditions (19)-(20) hold. Therefore, the FRB of 2-D system (15) can be obtained via the following theorem.

Theorem 3.2. *Given the system (15)-(16) and (c_1,c_2,I_0,R,d), fix $\alpha_l>0$, $0<\eta<1$, and find symmetric positive definite matrices P_l, Q_l, symmetric matrices M_l and positive scalars $\lambda_{l1},\lambda_{l2},\lambda_{l3}$ satisfying the LMIs (17), (18), (43), (44) and (45), where $l=1,2$. If the problem is feasible, the discrete system (15) is FRB with respect to (c_1,c_2,I_0,R,d), and the dynamic output feedback controller (11)-(12) with $L=Q^{-1}M$ solves the FRS problem of system (1)-(2).*

Example 3.1. In [30], Marszalek pointed out that some dynamical processes in gas absorption, water stream heating and air drying can be described by the Darboux equation:

$$\frac{\partial^2 s(x,t)}{\partial x \partial t} = a_1 \frac{\partial s(x,t)}{\partial t} + a_2 \frac{\partial s(x,t)}{\partial x} + a_0 s(x,t) + bf(x,t), \tag{46}$$

where $s(x,t)$ is an unknown function at x (space) $\in [0, x_f]$, and t (time) $\in [0, \infty]$, a_0, a_1, a_2 and b are real coefficients, and $f(x,t)$ is the input function.

In [31] the partial differential equation (PDE) model (46) was converted into a 2-D Roesser model of the form (1)-(2), where

$$A = \begin{bmatrix} 1 + a_1\Delta x & (a_1 a_2 + a_0)\Delta x \\ \Delta t & 1 + a_2 \Delta t \end{bmatrix}, \quad B = \begin{bmatrix} b\Delta x \\ 0 \end{bmatrix},$$

and $C = [C_1, C_2]$. Now let $a_0 = 37.9$, $a_1 = 2.7$, $a_2 = -12$, $b = -20$, $\Delta x = 0.2$, $\Delta t = 0.05$, $C_1 = 10$, $C_2 = 15$, then

$$A = \begin{bmatrix} 1.54 & 1.1 \\ 0.05 & 0.4 \end{bmatrix}, \quad B = \begin{bmatrix} -4 \\ 0 \end{bmatrix}, \quad C = [10, 15].$$

Suppose that $R = I$, $I_0 = I_1 \times I_2 = [0,5] \times [0,5]$, $c_1 = 0.7$, $c_2 = 20$, $d = 1$ and $x^h(0,j) = 0.76$, $x^v(i,0) = 0.26$.

When control input $u(i,j) = 0$, it is easy to check that the weighted-state $x^T(i,j)Ix^(i,j) > 20$, see Figure 1, then open-loop system is not FRS with $(0.7, 20, [0,5] \times [0,5], I)$ with the initial condition $x^h(0,j) = 0.76$, $x^v(i,0) = 0.26$.

In the following, we design a dynamic output feedback controller such that the closed-loop system is FRS.

First, we devise a state feedback controller K, which ensures the system $x^+(i,j) = (A + BK)x(i,j)$ FRS.

According to Theorem 3.3 in [25], let $c_1^{'} = 0.58$, $c_2^{'} = 0.07$, with $c_1^{'} + c_2^{'} < c_1 = 0.7$, $c^{'} = c_2 = 20$, $\eta = 0.9$, using LMI toolbox of Matlab, the conditions are feasible with $\alpha_1^{'} = 1.05$, $\alpha_2^{'} = 1.10$, $\beta_1^{'} = 35.5$, $\beta_2^{'} = 3.45$, and the state feedback controller is

$$K = [0.3850, 0.2750]. \tag{47}$$

Next, we design an observer gain L to guarantee the system (15) FRB.

By employing LMI control toolbox and Theorem 3.2, a feasible solution of the LMIs (17)-(18), (43)-(45) with $\eta = 0.9$, $\alpha_1 = 1.05$, $\alpha_2 = 1.06$ can be derived as follows

$$P = \begin{bmatrix} 7.1720 & 0 \\ * & 105.5938 \end{bmatrix}, \quad Q = \begin{bmatrix} 22.2962 & 0 \\ * & 32.3899 \end{bmatrix}, \quad M = \begin{bmatrix} -2.6483 \\ -0.5877 \end{bmatrix}.$$

Thus, we can obtain

$$L = Q^{-1}M = \begin{bmatrix} -0.1188 \\ -0.0181 \end{bmatrix}, \tag{48}$$

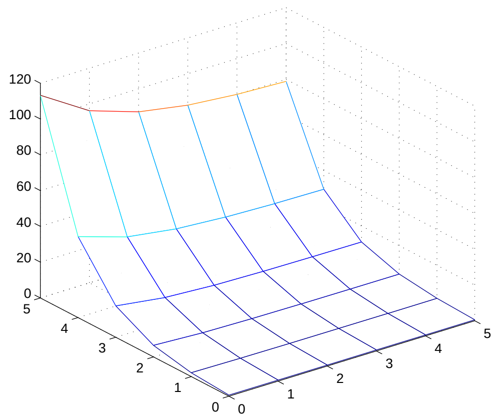

Figure 1: Weighted-state before stabilization

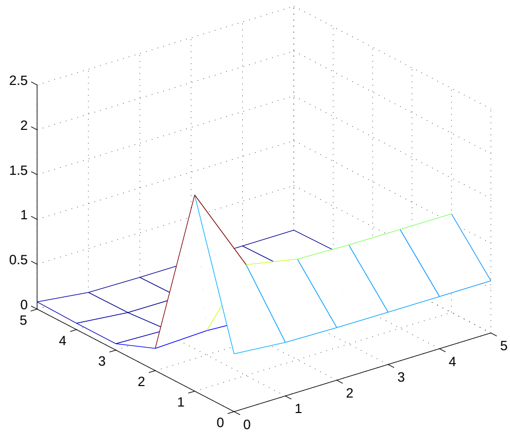

Figure 2: Weighted-state after stabilization

Hence the dynamic output feedback controller that stabilizes the system (1)-(2) in finite-region can be obtained as (7)-(8) with

$$A_c = \begin{bmatrix} -1.1880 & -1.7820 \\ -0.1310 & 0.1285 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0.1188 \\ 0.0181 \end{bmatrix}, \quad C_c = [0.3850, 0.2750], \quad D_c = 0.$$

Figure 2 shows the weighted-state $x^T(i,j)Rx(i,j)$ of closed-loop system with the initial condition $x^h(0,j) = 0.76$, $x^v(i,0) = 0.26$.

4 Conclusions

In this paper, the problem of finite-region stabilization for discrete 2-D Roesser models via dynamic output feedback has been studied. By designing a dynamic output feedback controller having a state feedback-observer structure, we get a closed-loop system that treats the state estimation errors as external perturbations. Then, the problem is translated into the problem for designing an observer to guarantee the closed-loop system FRB. Finally, we give two sufficient conditions for the existence of such a dynamic output feedback controller that guarantees the closed-loop system to be FRS. Further, this problem can also be studied in a similar way for other models of discrete 2-D systems.

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